Assignment 1

1.

To prove that M_i is a group, we need to demonstrate that M_i satisfies the four group properties: closure, associativity, identity element, and the existence of inverse elements.

(1) Closure:

By definition, we need to show that $\forall M_a, M_b \in \{M_i\}, M_a \times M_b = M_c \in \{M_i\}$,

$$\because M_a \times M_b = \begin{bmatrix} R_a & t_a \\ 0^T & 1 \end{bmatrix} \times \begin{bmatrix} R_b & t_b \\ 0^T & 1 \end{bmatrix} = \begin{bmatrix} R_a \times R_b & R_a \times t_b + t_a \\ 0^T & 1 \end{bmatrix} \tag{1}$$

$\because R_i \in \mathbb{R}^{3\times 3}$, and R_i is an orthogonal matrix, $R_i \times R_i^T = E$,

$\therefore R_a \times R_a^T = E, R_b \times R_b^T = E$

$$\begin{aligned} &\therefore (R_a \cdot R_b) \times (R_a \cdot R_b)^T \\ &= (R_a \cdot R_b) \times (R_b^T \cdot R_a^T) \\ &= R_a \cdot (R_b \times R_b^T) \cdot R_a^T \\ &= R_a \cdot R_a^T \\ &= E, \end{aligned}$$

$\therefore R_a \cdot R_b \in R_i$ is also an orthogonal matrix.

$\because t_i \in \mathbb{R}^{3\times 1}$,

$\therefore R_a \times t_b + t_a \in \mathbb{R}^{3\times 1}$,

$\therefore R_a \times t_b + t_a \in t_i$

$\therefore M_a \times M_b \in M_i$, and closure is established.

(2) Associativity:

By definition, we need to show that $\forall M_a, M_b, M_c \in \{M_i\}, (M_a \times M_b) \times M_c = M_a \times (M_b \times M_c)$,

$$\because (M_a \times M_b) \times M_c = \begin{bmatrix} R_a \times R_b & R_a \times t_b + t_a \\ 0^T & 1 \end{bmatrix} \times \begin{bmatrix} R_c & t_c \\ 0^T & 1 \end{bmatrix}$$

$$= \begin{bmatrix} R_a \times R_b \times R_c & (R_a \times R_b) \cdot t_c + R_a \times t_b + t_a \\ 0^T & 1 \end{bmatrix} \tag{2}$$

On the other hand,

$$\begin{aligned} \because M_a \times (M_b \times M_c) &= \begin{bmatrix} R_a & t_a \\ 0^T & 1 \end{bmatrix} \times \begin{bmatrix} R_b \times R_c & R_b \times t_c + t_b \\ 0^T & 1 \end{bmatrix} \\ &= \begin{bmatrix} R_a \times R_b \times R_c & R_a \times (R_b \times t_c + t_b) + t_a \\ 0^T & 1 \end{bmatrix} \\ &= \begin{bmatrix} R_a \times R_b \times R_c & R_a \times R_b \cdot t_c + R_a \times t_b + t_a \\ 0^T & 1 \end{bmatrix} \end{aligned} \tag{3}$$

It is clear that $(2) = (3)$, demonstrating associativity.

(3) Identity Element:

By definition, we need to show that $\exists M_e \in \{M_i\}, \forall M_a \in \{M_i\}, M_e \times M_a = M_a \times M_e = M_a$.
It is evident that $M_e = \begin{bmatrix} E & 0 \\ 0^T & 1 \end{bmatrix}$.

$\because M_a \times M_e = \begin{bmatrix} R_a & t_a \\ 0^T & 1 \end{bmatrix} \times \begin{bmatrix} E & 0 \\ 0^T & 1 \end{bmatrix}$

$= \begin{bmatrix} R_a \times E & R_a \times 0 + t_a \\ 0^T & 1 \end{bmatrix}$

$= \begin{bmatrix} R_a & t_a \\ 0^T & 1 \end{bmatrix} = M_a;$

$M_e \times M_a = \begin{bmatrix} E & 0 \\ 0^T & 1 \end{bmatrix} \times \begin{bmatrix} R_a & t_a \\ 0^T & 1 \end{bmatrix}$

$= \begin{bmatrix} E \times R_a & E \times t_a + 0 \\ 0^T & 1 \end{bmatrix}$

$= \begin{bmatrix} R_a & t_a \\ 0^T & 1 \end{bmatrix} = M_a,$

$\therefore$ There exists an identity element M_e.

(4) Existence of Inverse Elements:

By definition, we need to show that $\exists M_e \in \{M_i\}, \forall M_a \in \{M_i\}, \exists M_a^{-1} \in \{M_i\}, M_a^{-1} \times M_a = M_a \times M_a^{-1} = M_e$,

First, we calculate M_a^{-1}:

We assume $M_a \times M_b = M_e$, and from (3), we know that $M_e = \begin{bmatrix} E & 0 \\ 0^T & 1 \end{bmatrix}$, Comparing the elements, we get:

$$\begin{cases} R_a \times R_b = E \\ R_a \cdot t_b + t_a = 0 \end{cases}$$

Solving for R_b and t_b, we get:

$$\begin{cases} R_b = R_a^{-1} \\ t_b = -R_a^{-1} \cdot t_a \end{cases}$$

Now, substituting these values back, we can see that $M_b \times M_a = \begin{bmatrix} R_b \times R_a & R_b \times t_a + t_b \\ 0^T & 1 \end{bmatrix} = \begin{bmatrix} E & 0 \\ 0^T & 1 \end{bmatrix} = M_e$ is also satisfied.

$\because R_a \in R_i$, and it is an orthogonal matrix, and $t_a \in \mathbb{R}^{3\times 1}$

$\therefore R_b = R_a^{-1} = R_a^T$ exists, $t_b = -R_a^{-1} \cdot t_a$ also exists.Thus, $M_a^{-1} = \begin{bmatrix} R_a^{-1} & -R_a^{-1} \cdot t_a \\ 0^T & 1 \end{bmatrix}$, the existence of inverse elements for each group element is proven.

In conclusion, M_i is a group.

2.

a. Prove that the matrix M is positive semidefinite:

By definition, we need to prove that $\forall x \in \mathbb{R}^{n\times 1}$, $x^T \cdot M \cdot x \geq 0$, where x is a non-zero column vector.

It is known that M is a real symmetric matrix. Let $x = \begin{bmatrix} u \\ v \end{bmatrix}$,

$$\begin{aligned} &\therefore x^T \cdot M \cdot x \\ &= \textstyle\sum_{(x_i,y_i)\in w} \begin{bmatrix} u & v \end{bmatrix} \cdot \begin{bmatrix} I_x^2 & I_x \cdot I_y \\ I_x \cdot I_y & I_y^2 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} \\ &= \textstyle\sum_{(x_i,y_i)\in w} (u^2 I_x^2 + 2uvI_xI_y + v^2 I_y^2) \\ &= \textstyle\sum_{(x_i,y_i)\in w} (uI_x + vI_y)^2 \end{aligned}$$

$\because (uI_x + vI_y)^2 \geq 0$

$\therefore x^T \cdot M \cdot x \geq 0,$

$\therefore M$ is positive semi-definite.

b. Prove that when M is positive definite, the expression $\begin{bmatrix} x & y \end{bmatrix} \cdot M \cdot \begin{bmatrix} x \\ y \end{bmatrix}$ represents an ellipse:

Assuming the final result is $M = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$, and since M is positive definite, we have $a > 0$ and $c > 0$, as well as $ac - b^2 > 0$.

$\therefore \begin{bmatrix} x & y \end{bmatrix} \cdot M \cdot \begin{bmatrix} x \\ y \end{bmatrix} = ax^2 + 2bxy + cy^2$

$\therefore 4ac - (2b)^2 > 0$

The general equation of an ellipse is $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$,

By the definition of an ellipse equation, when $4AC - B^2 > 0$ and $A > 0, C > 0$, it represents an ellipse;

$\therefore \begin{bmatrix} x & y \end{bmatrix} \cdot M \cdot \begin{bmatrix} x \\ y \end{bmatrix}$ represents an ellipse.

c. Find the lengths of the major and minor axes of the ellipse:

From the given information, we know that M is positive definite, and its eigenvalues are λ_1 and λ_2, where $\lambda_1 > \lambda_2 > 0$.

It is known that the standard equation of an ellipse is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, where a and b are the lengths of the major and minor axes, respectively.

Next, we diagonalize M into $D = \operatorname{diag}(\lambda_1, \lambda_2)$ using the orthogonal matrix P:

$P^T MP = D$

We can express $\begin{bmatrix} x \\ y \end{bmatrix}$ as a linear combination with respect to P:

$\begin{bmatrix} x \\ y \end{bmatrix} = P \times \begin{bmatrix} x' \\ y' \end{bmatrix}$

$\therefore \begin{bmatrix} x & y \end{bmatrix} \cdot M \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x' & y' \end{bmatrix} \cdot (P^T MP) \cdot \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x' & y' \end{bmatrix} \cdot D \cdot \begin{bmatrix} x' \\ y' \end{bmatrix} = \lambda_1 x'^2 + \lambda_2 y'^2$

Now, the quadratic form matches the standard equation of an ellipse,
$\therefore a = \frac{1}{\sqrt{\lambda_1}},\ b = \frac{1}{\sqrt{\lambda_2}}$。

3.

It is evident that $A^T A$ is an $n \times n$ square matrix, and it is symmetric.

When $R(A) = n$, for $\forall y \in \mathbb{R}^{n \times 1}$ with $y \neq 0$, we have $Ay \neq 0$,

$\therefore y^T (A^T A) y = (Ay, Ay) = ||Ay||_2^2 > 0$

By the definition of a positive definite matrix, we have $A^T A$ as a positive definite matrix.

From the properties of positive definite matrices, we know that $det(A^T A) > 0$.

$\therefore A^T A$ is invertible.